ON THE DEBARRE-DE JONG AND BEHESHTI-STARR CONJECTURES ON HYPERSURFACES WITH TOO MANY LINES

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Abstract. We show that the Debarre-de Jong conjecture that the Fano scheme of lines on a smooth hypersurface of degree at most n in $\mathbb{P}^n$ must have its expected dimension, and the Beheshti-Starr conjecture that bounds the dimension of the Fano scheme of lines for hypersurfaces of degree at least n in $\mathbb{P}^n$, reduce to determining if the intersection of the top Chern classes of certain vector bundles is nonzero.

1. Introduction

Let K be an algebraically closed field of characteristic zero. Write X_{sing} for the singular points of a variety X, $\mathbb{P}^n = K\mathbb{P}^n$ and $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^n) = G(2, n+1)$ for the Grassmannian.

Conjecture 1.1. *Let $X^{n-1} \subset \mathbb{P}^n$ be a hypersurface of degree $d \geq n$ and let $\mathbb{F}(X) \subset \mathbb{G}(\mathbb{P}^1, \mathbb{P}^n)$ denote the Fano scheme of lines on X. Let $B \subset \mathbb{F}(X)$ be an irreducible component of maximal dimension. Let $\mathcal{I}_B := \{(x, E) \mid x \in X, E \in B, x \in \mathbb{P}E\}$ and let π, ρ denote the projections to X and B. Let $X_B = \pi(\mathcal{I}_B) \subseteq X$ and let $\tilde{\mathcal{C}}_x = \pi\rho^{-1}\rho\pi^{-1}(x)$.*

If $\dim \mathbb{F}(X) \geq n-2$, then for all $x \in X_B$, $\tilde{\mathcal{C}}_x \cap X_{sing} \neq \emptyset$.

By taking hyperplane sections in the case $d = n$, Conjecture 1.1 would imply the following conjecture, which was conjectured independently by O. Debarre and J. de Jong:

Conjecture 1.2 (Debarre-de Jong conjecture)**.** *Let $X^{n-1} \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$, then the dimension of the Fano scheme of lines on X equals $2n-d-3$.*

Our conjecture extends to smaller degrees as follows:

Conjecture 1.3. *Let $X^{n-1} \subset \mathbb{P}^n$ be a hypersurface of degree $n-l$. Let $B \subset \mathbb{F}(X)$ be an irreducible component of dimension $n-2$, with $\mathcal{I}_B, X_B$ etc... as above. If codim$(X_B, X) \geq l$ and $\mathcal{C}_x$ is reduced for general $x \in X_B$, then for all $x \in X_B$, $\tilde{\mathcal{C}}_x \cap X_{sing} \neq \emptyset$.*

The cases $X_B = X$ and $\operatorname{codim}(X_B, X) = \frac{n}{2}$ are known, e.g., they appear in Debarre's unpublished notes containing Conjecture 1.2. In [6], J. Harris et. al. proved Conjecture 1.2 when d is small with respect to n. Debarre in his unpublished notes also proved the case $d = n \leq 5$ and A. Collino [3] had earlier proven the case $d = n = 4$. In [1], R. Beheshti proved the case $d = n \leq 6$ and a different proof was also given in [7].

Conjecture 1.1 would also imply a special case of a conjecture of Beheshti and J. Starr (Question 1.3 of [2]), about $\mathbb{P}^k$'s on hypersurfaces, which, in the same paper, Beheshti proved for $k \geq (n-1)/4$ and Conjecture 1.1 would prove for $k = 1$.

Central to our work is finding additional structure on the tangent space to B at a general point. This structure gives rise to vector bundles on $\mathcal{C}_x$. We prove Conjecture 1.1 when the construction gives rise to exactly one vector bundle, see Theorem 3.6.

Landsberg supported by NSF grant DMS-0805782.

1.1. **Overview.** The statement of Conjecture 1.1 indicates how one should look for singular points. Say $y \in X$ and we want to determine if $y \in X_{\text{sing}}$, i.e., if $v_0, v_1, ..., v_n$ is a basis of W with $y = [v_0]$, and P an equation for X, we would need that all partial derivatives of local coordinates in y vanish. This is expressed by the n equations $dP_y(v_1) = \cdots = dP_y(v_n) = 0$. If we fix a line $\mathbb{P}E$ with $y \in \mathbb{P}E \subset X$ and look for a singular point of X on $\mathbb{P}E$, if e_1, e_2 is a basis of E that we expand to a basis $e_1, e_2, w_1, ..., w_{n-1}$ of W, the equations $dP_y(e_1) = dP_y(e_2) = 0$ come for free, so we have one less equation to satisfy.

A further simplification is obtained by a study of $T_EB \subset T_EG(2, W) = E^* \otimes W/E$. We observe T_EB is the kernel of the map $\alpha \otimes w \mapsto \alpha \circ (w \lrcorner P)|_E$, described in Proposition 2.1. Moreover, we identify the tangent space $T_E\mathcal{C}_x \subset T_EB \subset E^* \otimes W/E$ to the Fano scheme of B-lines through x as a subspace $\hat{x}^{\perp E} \otimes \Pi$, where $\Pi \subset W/E$ is independent of $x \in \mathbb{P}E$, see Proposition 2.2. In the same Proposition we remark that $E^* \otimes \Pi \subset T_EB$ is the intersection of T_EB with the locus of rank 1 homomorphisms in $T_EG(2, W) = E^* \otimes W/E$. As a consequence, $T_EB/(E^* \otimes \Pi)$ corresponds to a linear subspace of the space of $2 \times m$ matrices of constant rank two, for which there are normal forms. The normal forms allow us to reduce the number of equations defining the singular locus on a given line even further, see §3. Now the number of equations we reduce by will depend on the dimension of Π, but it is always bounded by $\dim \tilde{\mathcal{C}}_x$, where $\tilde{\mathcal{C}}_x$ is the cone swept by the lines of B passing through a general point x. For this reason, one expects to find at least a finite number of singular points of X lying on $\tilde{\mathcal{C}}_x$.

In the second part of the paper we show that $X_{\text{sing}} \cap \tilde{\mathcal{C}}_x$ as the zero locus of a section of a vector bundle, see equation (5). We determine certain positivity properties of the vector bundles we work with in Lemma 4.1, observe an elementary case where X must be singular (Theorem 3.2) and prove Theorem 3.6.

Acknowledgement. We thank IHES for providing us an outstanding environment for collaborating on this project and Johan de Jong for useful comments on earlier versions of this paper. The second author would like to thank Remke Kloosterman for discussions during the preparation of this paper.

2. The tangent space to B

In this section we study $\mathbb{P}^k$'s on an arbitrary projective variety $X \subset \mathbb{P}W$. Let W denote a vector space over an algebraically closed field K of characteristic zero. For algebraic subsets $Z \subset \mathbb{P}W$, we let $\hat{Z} \subset W$ denote the affine cone and $\mathbb{F}_k(Z) \subset G(k+1, W) = \mathbb{G}(\mathbb{P}^k, \mathbb{P}W)$ denotes the Fano scheme of $\mathbb{P}^k$'s on Z. Let $X \subset \mathbb{P}W$ be a variety. Let $B \subset \mathbb{F}_k(X)$ be an irreducible component. Let $\mathcal{I}_B := \{(x, E) \mid E \in B, x \in \mathbb{P}E\}$ be the incidence correspondence and let π, ρ denote the projections to X and B. Let $X_B = \pi(\mathcal{I}_B)$. Let $\mathcal{C}_x = \rho\pi^{-1}(x)$ and let $\tilde{\mathcal{C}}_x = \pi\rho^{-1}(\mathcal{C}_x)$, so $\tilde{\mathcal{C}}_x \subset X \subset \mathbb{P}W$ is a cone with vertex x and base isomorphic to $\mathcal{C}_x$.

For a vector space V, $v \in V$, and $q \in S^kV^*$, we let $v \lrcorner q \in S^{k-1}V^*$ denote the contraction. We also write $q(v^a, w^{k-a}) = q(v, ..., v, w, ..., w)$ etc. when we consider q as a multi-linear form. We denote the symmetric product by $\circ$, e.g., for $v, w \in V$, $v \circ w \in S^2V$. The following proposition is essentially a rephrasing of the discussion on p. 273 of [4]. We include a short proof for the sake of completeness.

Proposition 2.1. *Let $E \in \mathbb{F}_k(X)$, then $T_E\mathbb{F}_k(X) = ker\, \sigma_{(X,E)}$ where*

$$\begin{aligned} \sigma_{(X,E)} : T_EG(k+1, W) = E^* \otimes W/E \to\ & \oplus_d Hom(I_d(X), S^dE^*) \\ \alpha \otimes w \mapsto\ & \{P \mapsto \alpha \circ (w \lrcorner P)|_E\}. \end{aligned} \tag{1}$$

Proof. We first note that $(w \lrcorner P)|_E$ is well defined because $P|_E = 0$. Without loss of generality, we can restrict to the case where X is a hypersurface defined by a degree d polynomial P. The general case will follow by considering intersections.

Let $e_0,\dots,e_k$ be a basis of E, $\alpha_0,\dots,\alpha_k$ be the dual basis. A tangent vector $\eta=\alpha_0\otimes\bar{w}_0+\cdots+\alpha_k\otimes\bar{w}_k\in T_EG(k+1,W)$ corresponds to the first order deformation $E_t=\langle e_0+tw_0,\dots,e_k+tw_k\rangle$ of E in W, where the w_j are arbitrary liftings of the $\bar{w}_j$ to W. Recall that $E=\langle e_0,\dots,e_k\rangle$ belongs to $\mathbb{F}_k(X)$ if and only if P vanishes on all points of $\mathbb{P}E$, i.e., if and only if $P(e_0^{b_0},\dots,e_k^{b_k})=0$ for all $b_0,\dots,b_k$ such that $b_0+\cdots+b_k=d$. Therefore, the condition $\eta\in T_E\mathbb{F}_k(X)$ is equivalent to the vanishing of

$$\begin{aligned}&P((e_0+tw_0)^{b_0},(e_1+tw_1)^{b_1},...,(e_k+tw_k)^{b_k})\\&=P(e_0^{b_0},\dots,e_k^{b_k})+t[P(e_0^{b_0-1},w_0,e_1^{b_1},...,e_k^{b_k})+\cdots+P(e_0^{b_0},e_1^{b_1},...,e_k^{b_k-1},w_k)]\\&=0+t[(\alpha_0\circ P)(w_0,e_0^{b_0},\dots,e_k^{b_k})+(\alpha_1\circ P)(w_1,e_0^{b_0},\dots,e_k^{b_k})+\cdots+(\alpha_k\circ P)(w_k,e_0^{b_0},\dots,e_k^{b_k})]\\&=t[(\sigma_{(X,E)}(\eta))(e_0^{b_0},\dots,e_k^{b_k})]\end{aligned}$$

in $K[t]/(t^2)$ for every choice of $b_0,\dots,b_k$. This implies the claim. $\square$

Proposition 2.2. *Notations as above. Let x be a general point of X_B and E a general point of B with $x\in\mathbb{P}E$.*

(1) *If there exist $w\in W/E$ and $\alpha\in E^*\backslash 0$ such that $\sigma_{(X,E)}(\alpha\otimes w)=0$, then $E^*\otimes w\subset ker\,\sigma_{(X,E)}$.*
(2) *Letting $\Pi\subset W/E$ be maximal such that $E^*\otimes\Pi\subset ker\,\sigma_{(X,E)}$, then for $x\in\mathbb{P}E$, $T_E\mathcal{C}_x=\hat{x}^{\perp E}\otimes\Pi$.*

Proof. $\sigma_{(X,E)}(\alpha\otimes w)=0$ says $\alpha(u)P(w,u^{d-1})=0$ for all $u\in E$ and for all $P\in I(X)$. If $P(w,u^{d-1})=0$ for all u with $\alpha(u)\neq 0$, then $P(w,u^{d-1})=0$ for all $u\in E$, thus $E^*\otimes w\subset\ker\sigma_{(X,E)}$. The second assertion is clear. $\square$

3. How to find singular points on X

We now specialize to the case $k=1$ and X is a hypersurface in $\mathbb{P}^n=\mathbb{P}W$. In this case $T_EB/(E^*\otimes\Pi)$ is a linear subspace of $K^2\otimes K^m$ of constant rank two, where $m=n-1-\dim T_E\mathcal{C}_x$ in view of Proposition 2.2. There is a normal form for linear subspaces L of $K^2\otimes K^m$ containing no decomposable vectors. Namely for every basis α^1,α^2 of K^2, there exist a basis $w_1,...,w_m$ of K^m and integers $s_1,...,s_r$, with $r=m-\dim L$, $s_1+\cdots+s_r=m$ and $s_1\geq s_2\geq\cdots\geq s_r\geq 1$ such that

$$\begin{aligned}L=\langle&\alpha^1\otimes w_1-\alpha^2\otimes w_2,\alpha^1\otimes w_2-\alpha^2\otimes w_3,...,\alpha^1\otimes w_{s_1-1}-\alpha^2\otimes w_{s_1},\\&\alpha^1\otimes w_{s_1+1}-\alpha^2\otimes w_{s_1+2},\alpha^1\otimes w_{s_1+2}-\alpha^2\otimes w_{s_1+3},...,\alpha^1\otimes w_{s_2+s_1-1}-\alpha^2\otimes w_{s_2+s_1},\\&\dots\\&\alpha^1\otimes w_{s_{r-1}+\cdots+s_1+1}-\alpha^2\otimes w_{s_{r-1}+\cdots+s_1+2},...,\alpha^1\otimes w_{s_{r-1}+\cdots+s_1}-\alpha^2\otimes w_m\rangle\end{aligned}\tag{2}$$

The existence of this normal form is likely to be well known to the experts, although we could not find an explicit reference. The proof is left to the reader. Note that the normal form gives a basis of L divided into r blocks of length $s_1-1,\dots,s_r-1$. In particular, if for some index j we have $s_j=1$, then the corresponding block is empty.

Applying this normal form, we obtain a normal form for T_EB. Note that in this case $r=m-\dim T_EB/(E^*\otimes\Pi)=n-1-\dim T_EB+\dim T_E\mathcal{C}_x$. From now on, we will assume $\dim B\geq n-2$, so $r\leq\dim\mathcal{C}_x+1$, with equality holding generically if $\dim B=n-2$ and B is reduced.

Lemma 3.1. *Let $X\subset\mathbb{P}W$ be as above and assume $deg(X)=d\geq 1+s_1$. Let E be a general point of B. Then there exist $p_j^E\in S^{d-s_j}E^*$, $1\leq j\leq r$ such that*

$$Image\,\sigma_{(X,E)}=S^{s_1}E^*\circ p_1^E+\cdots+S^{s_r}E^*\circ p_r^E$$

where $w_1, ..., w_{n-1}$ *is a basis of* W/E *such that* $\Pi = \langle w_{m+1}, ..., w_{n-1}\rangle$ *and* $w_1, ..., w_m$ *are adapted to the normal form* (2)*.*

We remark that here and in Lemma 3.3 below, one can drop the assumption that E is a general point of B. The only change at special points is that the normal form (2) will be different.

Proof. Apply the normal form to $\ker \sigma_{(X,E)}/(E^* \otimes \Pi)$. For $1 \leq j \leq s_1 - 1$ we have

$$\alpha^1 \circ (w_j \lrcorner P)|_E = \alpha^2 \circ (w_{j+1} \lrcorner P)|_E \tag{3}$$

Since α^1, α^2 are linearly independent, for $j = 1$ this implies there exists $\phi_1 \in S^{d-2}E^*$ such that $(w_1 \lrcorner P)|_E = \alpha^2 \circ \phi_1$ and $(w_2 \lrcorner P)|_E = \alpha^1 \circ \phi_1$. But for the same reason, when $j = 2$ we see there exists $\phi_2 \in S^{d-3}E^*$ such that

$$(w_1 \lrcorner P)|_E = (\alpha^2)^2 \circ \phi_2, \quad (w_2 \lrcorner P)|_E = (\alpha^1 \circ \alpha^2) \circ \phi_2, \quad (w_3 \lrcorner P)|_E = (\alpha^1)^2 \circ \phi_2 \tag{4}$$

and so on until we arrive at $\phi_{s_1-1} =: p_1^E \in S^{d-s_1}E^*$, such that $(w_j \lrcorner P)|_E = (\alpha^1)^{j-1}(\alpha^2)^{s_1-j}p_1^E$ for $1 \leq j \leq s_1$. In particular, $S^{s_1}E^* \circ p_1^E \subset \mathrm{Image}\, \sigma_{(X,E)}$. Continuing in this way for the other chains in the normal form we obtain polynomials $p_1^E, ..., p_r^E$ with $S^{s_k}E^* \circ p_k^E \subset \mathrm{Image}\, \sigma_{(X,E)}$. Note that if $s_k = 1$ we set $p_k^E = (w_{s_{k-1}+\cdots+s_1+1} \lrcorner P)|_E$. $\square$

Note that without assumptions on the degree, the conclusion of Lemma 3.1 can fail, e.g., if $d = 3$ and $s_1 = m = 3$, as in the case of a general cubic hypersurface, then (4) only says $(w_1 \lrcorner P)|_E = (\alpha^2)^2$, $(w_2 \lrcorner P)|_E = \alpha^1 \circ \alpha^2$ and $(w_3 \lrcorner P)|_E = (\alpha^1)^2$. This does imply that the image of $\mathbb{P}E$ under the Gauss map of X is a rational normal curve of degree two in $\mathbb{P}(E+\Pi)^\perp \subset \mathbb{P}W^*$ and one can obtain similar precise information about the Gauss image of $\mathbb{P}E$ in other cases. However, as long as $\deg(X) \geq s_1 + 1$ the conclusion holds.

When $s_1 = n - 1 - \dim \mathcal{C}_x$ there is a single polynomial on $\mathbb{P}E$ whose zero set corresponds to singular points of X. Thus:

Theorem 3.2. *Let* $X^{n-1} \subset \mathbb{P}^n$ *be a hypersurface, with* $B, \tilde{\mathcal{C}}_x$ *etc. as above. If* $deg(X) \geq s_1 + 1$ *and* $s_1 = n - 1 - dim\mathcal{C}_x$*, then for all* $E \in B$*,* $\mathbb{P}E \cap X_{sing} \neq \emptyset$*.*

Lemma 3.3. *Let* X *be as above and let* E *be a general point of* B*. Write* $\{deg\ p_k^E : 1 \leq k \leq r\} = \{\delta_1 < \delta_2 < \cdots < \delta_c\}$ *and set* $i_j = \#\{p_k^E : deg\ p_k^E \leq \delta_j\}$ *for all* $j \leq c$*. Note that if* $s_r = 1$ *in the normal form* (2)*, then* $i_{c-1} = \#\{k : s_k > 1\}$ *and* $i_c = r$*. Consider the vector spaces*

$$\begin{aligned}
\hat{M}_1 = M_1 &:= \langle p_1^E, ..., p_{i_1}^E\rangle \subset S^{\delta_1}E^* \\
\hat{M}_2 &:= \langle p_{i_1+1}^E, ..., p_{i_2}^E, \hat{M}_1 \circ S^{\delta_2-\delta_1}E^*\rangle \subset S^{\delta_2}E^* \\
M_2 &:= \hat{M}_2/(\hat{M}_1 \circ S^{\delta_2-\delta_1}E^*) \subset S^{\delta_2}E^*/(\hat{M}_1 \circ S^{\delta_2-\delta_1}E^*) \\
&\vdots \\
\hat{M}_{c-1} &:= \langle p_{i_{c-2}+1}^E, ..., p_{i_{c-1}}^E, \hat{M}_{c-2} \circ S^{\delta_{c-1}-\delta_{c-2}}E^*\rangle \subset S^{\delta_{c-1}}E^* \\
M_{c-1} &:= \hat{M}_{c-1}/(\hat{M}_{c-2} \circ S^{\delta_{c-1}-\delta_{c-2}}E^*) \subset S^{\delta_{c-1}}E^*/(\hat{M}_{c-2} \circ S^{\delta_{c-1}-\delta_{c-2}}E^*) \\
\hat{M}_c &:= \langle p_{i_{c-1}+1}^E, ..., p_{i_c}^E, \hat{M}_{c-1} \circ S^{\delta_c-\delta_{c-1}}E^*\rangle \subset S^{\delta_c}E^* \\
M_c &:= \hat{M}_c/(\hat{M}_{c-1} \circ S^{\delta_c-\delta_{c-1}}E^*) \subset S^{\delta_c}E^*/(\hat{M}_{c-1} \circ S^{\delta_c-\delta_{c-1}}E^*)
\end{aligned}$$

These spaces are well defined and depend only on X, E*.*

The lemma is an immediate consequence of the uniqueness of the normal form up to admissible changes of bases. Let $I_E \subset Sym(E^*)$ denote the ideal generated by the $\hat{M}_j$. Note that the number of polynomials generating I_E is at most $\dim \mathcal{C}_x + 1$, independent of the normal form

(and $\dim \mathcal{C}_x + 1$ is the expected number of generators). Let $B' \subset B$ denote the Zariski open subset where the normal form is the same as that of a general point.

Proposition 3.4. *Let $E \in B'$ and let $[y] \in \mathbb{P}E$ be in the zero set of I_E, then $[y] \in X_{sing}$.*

Proof. $[y] \in X_{\rm sing}$ says that for all $w \in W$, $(w \lrcorner P)(y) = 0$. Let $w_1, \ldots, w_{n-1}$ be elements of W that descend to give a basis of W/E. Since $(u \lrcorner P)|_E = 0$ holds for all $u \in E$, the polynomial $(w \lrcorner P)|_E \in S^{d-1}E^*$ is a linear combination of the $(w_i \lrcorner P)|_E$. As each $(w_i \lrcorner P)|_E$ contains one of the p_j^E as a factor, the hypothesis implies $w \lrcorner P$ vanishes at y. $\square$

We now allow E to vary. Let $\mathcal{S} \to G(2, W)$ denote the tautological rank two subspace bundle and note that the total space of $\mathbb{P}\mathcal{S}|_B$ is our incidence correspondence $\mathcal{I}_B$. Since all calculations are algebraic, M_1 gives rise to a rank i_1 algebraic vector bundle $\mathcal{M}_1 \subset S^{\delta_1}\mathcal{S}^*|_{B'}$, and M_2 gives rise to a rank $i_2 - i_1$ algebraic vector bundle $\mathcal{M}_2 \subset ((S^{\delta_2}\mathcal{S}^*)/(\mathcal{M}_1 \circ S^{\delta_2-\delta_1}\mathcal{S}^*))|_{B'}$, etc. . . finally giving a bundle of ideals $\mathcal{I} \subset Sym(\mathcal{S}^*)|_{B'}$.

Now, since Grassmannians are compact, along any curve E_t in B with $E_t \in B'$ for $t \neq 0$, we have well defined limits as $t \to 0$, and thus we may define $\mathbf{I}_0^{E_t} \subset Sym(E_0^*)$. Note that if we approach E_0 in different ways, we could obtain different limiting ideals, nevertheless we have:

Proposition 3.5. *Let $E \in B$ and let $[y] \in \mathbb{P}E$ be in the zero set of $\mathbf{I}_0^{E_t}$, then $[y] \in X_{sing}$.*

Proof. Although this is a standard argument, we give details in a special case to show that at points of $B \backslash B'$ the situation is even more favorable. Work locally in a coordinate patch. First note that we may choose a fixed $\alpha^1, \alpha^2 \in W^*$ that restrict to a basis of E^* for all E in our coordinate patch and still obtain the normal form by linear changes of bases in W/E. So along our curve E_t we consider α^1, α^2 and $w_1^t, ..., w_{n-1}^t$ such that for $t \neq 0$ (and small), $\Pi = \langle w_{m+1}^t, ..., w_{n-1}^t \rangle$ and we have a fixed normal form, e.g., say $\alpha^1 \otimes w_1^t - \alpha^2 \otimes w_2^t, \alpha^1 \otimes w_3^t - \alpha^2 \otimes w_4^t \in \ker \sigma_t$ for all small t, giving rise to polynomials ϕ_t, ψ_t such that

$$w_1^t \lrcorner P|_{E_t} = \alpha^2 \circ \phi_t, \quad w_2^t \lrcorner P|_{E_t} = \alpha^1 \circ \phi_t, \quad w_3^t \lrcorner P|_{E_t} = \alpha^2 \circ \psi_t, \quad w_4^t \lrcorner P|_{E_t} = \alpha^1 \circ \psi_t.$$

In the limit, we may not assume that $w_1^0, ..., w_m^0$ are linearly independent.

First notice that if $\psi_0 = \mu\phi_0$, then although we have a well defined plane $\lim_{t \to 0}[\phi_t \wedge \psi_t]$ (which equals $[\phi_0 \wedge (\psi_0' - \mu\phi_0')]$ if $\phi_0 \wedge (\psi_0' - \mu\phi_0') \neq 0$), the vanishing of ϕ_0 already implies $[y] \in X_{\rm sing}$, as long as $w_1^0, ..., w_4^0$ are linearly independent.

Now consider the case we have a relation $l^1 w_1^0 + \cdots + l^4 w_4^0 = 0$. This implies we have a relation

$$\begin{aligned} 0 &= l_1\alpha^2 \circ \phi_0 + l^2\alpha^1 \circ \phi_0 + l^3\alpha^2 \circ \psi_0 + l^4\alpha^1 \circ \psi_0 \\ &= \alpha^1 \circ (l^2\phi_0 + l^4\psi_0) + \alpha^2 \circ (l^1\phi_0 + l^4\psi_0) \end{aligned}$$

Which implies (assuming all coefficients nonzero) $\psi_0 = \mu\phi_0$ with $\mu = -l^2/l^4 = -l^1/l^3$. In particular, the relation among the w_j^0 was not arbitrary. We also see that

$$(l^1 w_1^{0\prime} + \cdots + l^4 w_4^{0\prime}) \lrcorner P|_{E_0} = (\alpha^1 + \mu\alpha^2)(l^2\phi_0{}' + l^4\psi_0{}')$$

That is, assuming $z := (l^1 w_1^{0\prime} + \cdots + l^4 w_4^{0\prime})$ is linearly independent of $w_1^0, ..., w_4^0$, we obtain that $\mathbf{I}_0^{E_t}$ includes $z \lrcorner P|_{E_0}$. Otherwise, just differentiate further. $\square$

We would like to work with vector bundles over our entire space, which can be achieved by considering the product of Grassmann bundles $G(\text{rank}\, \hat{M}_1, S^{\delta_1}\mathcal{S}^*) \times \cdots \times G(\text{rank}\, \hat{M}_c, S^{\delta_c}\mathcal{S}^*) \to B$. Over $B' \subset B$ we have a well defined section of this bundle. Using the compactness of the Grassmannian and the limiting procedure described above, we extend these sections to obtain a space $\tau : \mathcal{B} \to B$, with fiber over points of B' a single point. Thus each M_j (resp. $\hat{M}_j$) gives

rise to a well defined vector bundle $\mathbf{M}_j \to \mathcal{B}$ (resp. $\hat{\mathbf{M}}_j \to \mathcal{B}$), and we have the corresponding bundle of ideals $\mathbf{I} \subset \tau^*(Sym(\mathcal{S}^*))$.

Let $\mathbf{S} = \tau^*(\mathcal{S})$ and $\mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta) = \tilde{\tau}^*(\mathcal{O}_{\mathbb{P}\mathcal{S}}(\delta))$, where $\tilde{\tau} : \mathbf{S} \to \mathcal{S}$ is the lift of τ. Consider the projection $q : \mathbb{P}(\mathbf{S}) \to \mathcal{B}$ and the bundles

$$q^*(\mathbf{M}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j)$$

Then $q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1) = q^*(\hat{\mathbf{M}}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1)$ has a canonical section $\mathbf{s}_1$ whose zero set $Z_1 \subset \mathbb{P}(\mathbf{S})$ is the zero set of $(\mathbf{I})_{\delta_1}$. For each $2 \leq j \leq c$, the corresponding bundle $q^*(\hat{\mathbf{M}}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j)$, has a canonical section $\hat{\mathbf{s}}_j$, whose zero set $Z_j \subset \mathbb{P}(\mathbf{S})$ is the zero set of $(\mathbf{I})_{\delta_j}$.

Fix a general point $x \in X_B$, let $\mathbf{C}_x = \tau^{-1}(\mathcal{C}_x) \subset \mathcal{B}$. The essential observation is that $\dim \tilde{\mathcal{C}}_x \geq r = \sum_j \operatorname{rank} \mathbf{M}_j$, so we expect $Z_c \cap q^{-1}(\mathbf{C}_x)$ to be nonempty. This would imply the existence of singular points, because the image of Z_c in X_B is contained in X_{sing}.

In more detail, we have a sequence of vector bundles $q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1), ..., q^*(\mathbf{M}_c)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_c)$ over $\mathbb{P}(\mathbf{S})$, whose ranks add up to r, such that $q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1)$ is equipped with a canonical section $\mathbf{s}_1$, and restricted to its zero set Z_1, $q^*(\mathbf{M}_2)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_2)$ has a canonical section $\mathbf{s}_2$, etc... such that if everything were to work out as expected, the zero set Z_c of $\mathbf{s}_c$, which is defined as a section of $q^*(\mathbf{M}_c)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_c)$ over Z_{c-1}, would have codimension r, which is the dimension of $\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$. Thus we expect $Z_c \cap \mathbb{P}(\mathbf{S})|_{\mathbf{C}_x} \neq \emptyset$, which would imply that $\tilde{\mathcal{C}}_x \cap X_{\text{sing}} \neq \emptyset$. Note that a sufficient condition for this is

$$c_{\text{top}}(q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1)) \cdot c_{\text{top}}(q^*(\mathbf{M}_2)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_2)) \cdots c_{\text{top}}(q^*(\mathbf{M}_c)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_c)) \neq 0, \tag{5}$$

where the intersection takes place in the Chow group of codimension r cycles on $\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$.

We were not able to prove this in general, but we are able to show:

Theorem 3.6. *The zero set of the canonical section of $\hat{\mathbf{M}}_1^* \otimes \tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_1))|_{\mathbf{C}_x}$ is always at least of the expected dimension.*

Another natural case to consider is the case where the $\mathbf{M}_j$ are all line bundles. For instance, consider the even further special case where there is just $\mathbf{M}_1, \mathbf{M}_2$ and both are line bundles. This case splits into two sub-cases, based on whether or not the zero section of $\mathbf{s}_1$ surjects onto all of X_B or not. In §6, we show that if $Z(\mathbf{s}_1)$ fails to surject onto X_B, then Conjecture 1.1 indeed holds.

Since $q^*(\mathbf{M}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j)$ only has a section defined over Z_{j-1}, it will be more convenient to work with the bundles $q^*(\hat{\mathbf{M}}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j)$ which have everywhere defined sections $\hat{\mathbf{s}}_j$.

The best situation for proving results about sections of bundles is when the bundles are ample, which fails here. However, below we show that if x is sufficiently general, the bundles $\hat{\mathbf{M}}_j^* \otimes \tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))|_{\mathbf{C}_x}$ are *generically ample.*

4. Generic ampleness

Recall ([5], Example 12.1.10) that a vector bundle $\mathcal{E}$ over a variety X is *generically ample* if it is generated by global sections and the canonical map $\mathbb{P}\mathcal{E}^* \to \mathbb{P}(H^0(X, \mathcal{E})^*)$ is generically finite. The locus where it is not finite is called the *disamplitude locus* $\operatorname{Damp}(\mathcal{E})$. In particular, if $Y \subset X$ is a subvariety such that $\mathcal{E}|_Y$ has a trivial quotient sub-bundle, then $Y \subset \operatorname{Damp}(\mathcal{E})$.

Generically ample bundles of rank $r \leq \dim X$ have the property that $c_1(\mathcal{E}), ..., c_r(\mathcal{E})$ are all positive, in the sense that their classes in the Chow group of X are linear combinations of effective classes with nonnegative coefficients, not all equal to 0.

Lemma 4.1. *For general $x \in X_B$, the bundles $\hat{\mathbf{M}}_j^* \otimes \tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))|_{\mathbf{C}_x}$ are generically ample.*

Proof. First, global generation is clear, as for all the $\hat{\mathbf{M}}_j$ we have a surjective map

$$S^{\delta_j}\mathbf{S}\otimes\tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))|_{\mathbf{C}_x}\to\hat{\mathbf{M}}_j^*\otimes\tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))|_{\mathbf{C}_x}.$$

Now take any choice of splitting $W=\hat{x}\oplus W'$, so the left hand side becomes $\tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}\oplus\mathcal{O}_{\mathbb{P}(W/\hat{x})}(1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))$ restricted to $\mathbf{C}_x$. Each factor is globally generated and this of course remains true when restricting to subvarieties.

The locus where the canonical map

$$\mathbb{P}(\oplus_{i=0}^{\delta_j}\mathcal{O}_{\mathbb{P}(W/\hat{x})}(-i))\to\mathbb{P}(H^0(\mathbb{P}(W/\hat{x}),\oplus_{i=0}^{\delta_j}\mathcal{O}_{\mathbb{P}(W/\hat{x})}(i))^*)$$

is not finite is the $\mathbb{P}\mathcal{O}_{\mathbb{P}(W/\hat{x})}$ factor. Hence, when we restrict to $\mathcal{C}_x$ and pull-back to $\mathbf{C}_x$, $\mathrm{Damp}(\hat{\mathbf{M}}_j^*\otimes\tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))|_{\mathbf{C}_x})$ is contained in the union of the following two loci:

- the locus where the map $\tau\colon\ \mathbf{C}_x\to\mathcal{C}_x$ has positive-dimensional fibers;
- the projection to $\mathbf{C}_x$ of the locus where the image of

$$\mathbb{P}(\hat{\mathbf{M}}_j\otimes\tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(-\delta_j))|_{\mathbf{C}_x})\to\mathbb{P}(\oplus_{i=0}^{\delta_j}\tau^*\mathcal{O}_{\mathbb{P}(W/\hat{x})}(-i))$$

  intersects $\mathbb{P}(\tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}))$.

The lemma will follow from Lemma 4.2 below and the fact that the general fiber of $\mathbf{C}_x\to\mathcal{C}_x$ is finite if x is general. Note that the image of $\mathbb{P}\left(\hat{\mathbf{M}}_j\otimes\tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(-\delta_j))|_{\mathbf{C}_x}\right)$ inside $\mathbb{P}(\oplus_{i=0}^{\delta_j}\tau^*\mathcal{O}_{\mathbb{P}(W/\hat{x})}(-i))$ intersects $\mathbb{P}(\tau^*\mathcal{O}_{\mathbb{P}(W/\hat{x})}|_{\mathbf{C}_x})$ precisely over the points $E\in\mathbf{C}_x$ such that the fiber $\hat{\mathbf{M}}_{j,E}$ contains a nonzero polynomial vanishing at x with multiplicity δ_j. ☐

Lemma 4.2. *For general $x\in X_B$ and general $E\in\mathcal{C}_x$, all nonzero elements $P\in(\mathbf{I}_E)_k$ vanish at x with multiplicity at most $k-1$ for any integer $k\le\delta_c$.*

Proof. Fix $E\in B$. Then the locus

$$\{[P]\in\mathbb{P}((\mathbf{I}_E)_k)\ \mid\ P=f^k\text{ for some }f\in E^*\}$$

is the intersection of $\mathbb{P}((\mathbf{I}_E)_k)$ with a degree k rational normal curve contained in $\mathbb{P}(S^kE^*)$. Hence, it consists of at most a finite number of points $[P_1],\dots,[P_R]$. Thus it suffices to choose a point $x\in\mathbb{P}E$ such that $P_j(x)\neq 0$ for all $j=1,\dots,R$. ☐

5. Proof of Theorem 3.6

Lemma 3.6 is a consequence of Lemma 4.1 for $j=1$, combined with the following lemma with $M=\hat{\mathbf{M}}_1|_{\mathbf{C}_x}$.

Lemma 5.1. *Let $M\subset S^p\mathbf{S}^*|_{\mathbf{C}_x}$ be a vector bundle such that $M^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$ is generically ample. Then the zero locus of the canonical section of $q^*M^*\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_{\mathbf{C}_x}}(p)$ is of dimension at least $\dim\mathbf{C}_x+1-\mathrm{rank}(M)$.*

The proof of Lemma 5.1 follows by several reductions which reduce the question to a basic fact about intersections on nontrivial $\mathbb{P}^1$-bundles over a curve:

Lemma 5.2. *Let $p_\xi\colon\ S\to\xi$ be a $\mathbb{P}^1$-bundle over a curve ξ, with a section $e\colon\ \xi\to S$ of negative self-intersection. If $\tilde{D}_1,\tilde{D}_2$ are effective divisors of S not contained in the image of e such that the restriction of p_ξ to each of them is finite, then $\tilde{D}_1\cap\tilde{D}_2\neq\emptyset$.*

Proof. The Picard group of S is generated by the class ξ_0 of the image of e and the class F of a fiber of p_ξ. Since S is not a product, one has $F^2=0$, $\xi_0\cdot F=1$ and $\xi_0^2=-k$ with k a positive integer. Choose irreducible components D_1, D_2 of the divisors, different from the image of e. Then $D_i=a_i\xi_0+b_iF$ with $a_i\ge 1$ (since it is the degree of $p_\xi|_{D_i}$) and $D_i\cdot\xi_0=b_i-a_ik\ge 0$. Then $D_1\cdot D_2=-a_1a_2k+a_1b_2+a_2b_1\ge a_1a_2k>0$. From this the claim follows. ☐

The proof of Lemma 5.1 relies on the following Lemma:

Lemma 5.3. *Let $M \subset S^p\mathbf{S}^*|_{\mathbf{C}_x}$ be a vector bundle such that $M^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$ is generically ample. Let $W' \subset W$ be any hyperplane not containing $\hat{x}$, set $H' = \mathcal{C}_x \cap \mathbb{P}W'$, and let $H \subset \mathbb{P}(\mathbf{S}|_{\mathbf{C}_x})$ be the preimage of H' under $\tau\colon \mathbb{P}(\mathbf{S}|_{\mathbf{C}_x}) \to \tilde{\mathcal{C}}_x$. Let $\mathbf{s}_M$ denote the canonical section of $q^*M^*\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_{\mathbf{C}_x}}(p)$.*

Then the intersection $Z(\mathbf{s}_M)\cap H$ has dimension at least $\dim \mathbf{C}_x - rank(M)$. In particular, it is nonempty if $rank\, M \le \dim \mathbf{C}_x$.

Proof. Consider the section $s_{M,W'} \in H^0(H', q^*\otimes\mathcal{O}_{\mathbb{P}(\mathbf{S}|_{\mathbf{C}_x})}(p))$ obtained by restricting $\mathbf{s}_M$ to H'. Then we have $Z(\mathbf{s}_M)\cap H = Z(s_{M,W'})$.

Observe that $\rho\colon \tilde{\mathcal{C}}_x \to \mathcal{C}_x$ and $q\colon \mathbb{P}(\mathbf{S}|_{\mathbf{C}_x}) \to \mathbf{C}_x$ become isomorphisms when restricted to, respectively, H' and H. In particular, since H' was an hyperplane section of $\mathcal{C}_x$, the isomorphism $H \cong \mathbf{C}_x$ so obtained induces an isomorphism $\mathcal{O}_{\mathbb{P}(\mathbf{S}|_{\mathbf{C}_x})}(1)|_H \cong \mathcal{O}_{\mathbf{C}_x}(1)$. Since the isomorphism $H \cong \mathbf{C}_x$ also induces an isomorphism $(q^*M)|_H \cong M$, one can view $s_{M,W'}$ as a global section of $M^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$. Therefore, if $Z(s_{M,W'}) \subset H$ is nonempty, it has codimension at most $\operatorname{rank} M$ in H [5, Prop. 14.1b]. It remains to show $Z(s_{M,W'}) \neq \emptyset$ if $\operatorname{rank} M \le \dim \mathbf{C}_x$.

Recall from [5, §14.1] that there is a localized Chern class associated to the section $s_{M,W'}$, which is a class in the Chow group of $Z(s_{M,W'})$ whose pull-back under the inclusion $Z(s_{M,W'}) \to \mathcal{C}_x$ is the top Chern class of $M^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$. Since $M^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$ is generically ample and of rank $\le \dim \mathbf{C}_x$, its top Chern class is positive. So the Chow group of $Z(s_{M,W'})$ contains a nontrivial class, and in particular $Z(s_{M,W'})$ cannot be empty. $\square$

Proof of Lemma 5.1. For every $E \in \mathbf{C}_x$, consider $N_E := (S^{p-1}E^*\circ\hat{x}^\perp)\cap M_E$, the linear subspace of M_E of forms vanishing on the point x. Without loss of generality, when E varies N_E gives rise to a vector subbundle $N \subset M$ of codimension 1. Indeed, if it were not so, there would be a point $E \in \mathbf{C}_x$ such that $N_E = M_E$, and then (E, x) would be a point of the zero locus of the canonical section, thus implying the claim.

We have an exact sequence $0 \to N \to M \to L \to 0$ where L is the quotient line bundle. Since $q^*N^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$ is a corank 1 quotient of $q^*M^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$, we can apply Lemma 5.3 to it. Hence, the zero locus of the canonical section of $q^*N^*\otimes\mathcal{O}_{\mathbf{C}_x}(p)$ contains an irreducible component Z which intersects all subvarieties $H \subset \mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$ which come from preimages of general hyperplane sections of $\tilde{\mathcal{C}}_x$.

Without loss of generality, we may assume that Z is of dimension 1, and that $q' := q|_Z :\ Z \to q(Z) =: \xi$ is a finite surjective map. Recall that the group of Weil divisors (up to numerical equivalence) of the ruled surface $\mathbb{P}(\mathbf{S})|_\xi$ is generated by the class ξ_0 of the tautological section of q' (i.e., $(\xi_0)_E = (E, x)$) and the class F of a fiber of q. From the effectivity of Z and from Lemma 5.3 we obtain $Z\cdot F \ge 1$, $Z\cdot\xi_0 \ge 0$.

To prove the claim, it suffices to show $Z\cdot c_1(q^*L^*|_\xi\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)) > 0$. We have $c_1(q^*L^*|_\xi\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p))\cdot F = d$ because $c_1(q^*L^*\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p))\cdot F = c_1(q^*L^*)\cdot F + c_1(\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p))\cdot F = 0 + p = p$. Recall that the canonical section of $q^*N^*|_\xi\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)$ vanishes on ξ_0 by construction. Therefore, the canonical section of $q^*M^*|_\xi\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)$ induces a section s_L of $q^*L^*|_\xi\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)$ on ξ_0. Since $N_E \subsetneq M_E$ for every $E \in \mathbf{C}_x$, we have that s_L cannot vanish identically on ξ_0. Hence $c_1(q^*L^*|_\xi\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p))\cdot\xi_0 \ge 0$, because it is the class of $Z(s_L)$ on ξ_0. Then the asserted inequality follows from Lemma 5.2 because $c_1(q^*L^*|_\xi\otimes\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p))$ is linearly equivalent to an effective divisor satisfying the hypotheses of Lemma 5.2. $\square$

6. Two line bundles

In this section, we prove the following result, which was announced in section 3.

Lemma 6.1. *Assume* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are line bundles, and that the projection* $Z(\mathbf{s}_1) \to X_B$ *is not surjective. Then for every* $x \in X_B$*, the zero set of* $\mathbf{s}_2|_{\mathbf{C}_x}$ *is of codimension at most* 2 *in* $\mathbf{C}_x$*.*

Therefore, we assume $\mathbf{M}_1$ and $\mathbf{M}_2$ are both line bundles.

As in the arguments above, it will be sufficient to work with a general point $x \in X_B$ and a sufficiently general irreducible curve $\xi \subseteq \mathbf{C}_x$ and show that the zero set of $\hat{\mathbf{s}}_2$ restricted to $\mathbb{P}(\mathbf{S})|_\xi$ is nonempty. The proof is based on showing that $Z(\hat{\mathbf{s}}_2) \cap \mathbb{P}(\mathbf{S})|_\xi$ coincides with the zero set of the canonical section of $(q|_\xi)^* N^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_2)$ a rank 2 vector bundle $N \subset S^{\delta_2}\mathbf{S}|_\xi$ satisfying the hypotheses of Lemma 5.1. We construct N under the assumption that the zero set $Z(\mathbf{s}_1)$ does not intersect the tautological section of $\mathbb{P}(\mathbf{S})|_\xi \to \xi$.

Since $\mathbf{M}_1$ is a line bundle, we have that $Z(\mathbf{s}_1) \subset \mathbb{P}(\mathbf{S})$ intersects every fiber of $\mathbb{P}(\mathbf{S}) \to \mathcal{B}$ in δ_1 points, counted with multiplicity. This follows from the very construction of the canonical section $\mathbf{s}_1$.

Without loss of generality in the choice of x and ξ, we may assume:

1. $\mathcal{O}_{\mathbf{C}_x}(1)$ restricts to a generically ample line bundle $\mathcal{O}_\xi(1)$ on ξ.
2. $Z := Z(\hat{\mathbf{s}}_1) \cap \mathbb{P}(\mathbf{S})|_\xi$ is not contained in the tautological section $\xi \to \mathbb{P}(\mathbf{S})|_\xi$.
3. $\hat{\mathbf{M}}_2^* \otimes \mathcal{O}_{\mathbf{C}_x}(\delta_2)$ is generically ample when restricted to ξ.
4. the map $q|_Z \colon Z \to \xi$ is finite.

The first assumption follows from the fact that $\mathbf{C}_x \to \mathcal{C}_x$ is generically finite, so $\mathbf{C}_x \not\subset \mathrm{Damp}(\mathcal{O}_{\mathbf{C}_x}(1))$ and the same holds for a generical $\xi \subset \mathbf{C}_x$. Assumption (2) follows from the genericity of x, and (3) follow from Lemma 4.1. Finally, if (4) did not hold, $Z(\hat{\mathbf{s}}_2)$ would contain δ_2 points on every 1-dimensional fiber of $q|_Z$ (counted with multiplicity), thus showing $Z(\hat{\mathbf{s}}_2) \neq \emptyset$.

For the rest of this section, we will often omit the restriction to ξ from our notation.

Recall we have short exact sequence:

$$0 \to S^{\delta_2 - \delta_1}\mathbf{S}^* \circ \mathbf{M}_1 \to \hat{\mathbf{M}}_2 \to \mathbf{M}_2 \to 0.$$

As a consequence, the section $\hat{\mathbf{s}}_2 \in H^0(\mathbb{P}(\mathbf{S}), q^*\hat{\mathbf{M}}_2^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_2))$ canonically induces a section $s \in H^0(Z, q^*\mathbf{M}_2^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_2))$. Assume that $Z(\hat{\mathbf{s}}_2) = \emptyset$, i.e., $Z(s) = \emptyset$ on Z. Then s induces a trivialization $q^*\mathbf{M}_2^*|_Z \otimes \mathcal{O}_Z(\delta_2) \cong \mathcal{O}_Z$.

In this set-up, Lemma 6 is equivalent to the following lemma:

Lemma 6.2. *Assume* Z *does not intersect the image of the tautological section* $s_0 \colon \xi \to \mathbb{P}(\mathbf{S})|_\xi$*. Then* $Z(\hat{\mathbf{s}}_2) \cap \mathbb{P}(\mathbf{S})|_\xi \neq \emptyset$*.*

Proof. Assume by contradiction $Z(\hat{\mathbf{s}}_2) \cap \mathbb{P}(\mathbf{S})|_\xi$ is empty. Fix a line $E \in \xi$. The fiber $\hat{\mathbf{M}}_{2,E}$ is spanned by all degree δ_2 multiples of polynomials in $\mathbf{M}_{1,E}$ and by an additional polynomial ϕ, which does not vanish on any point of Z.

Recall that no nonzero polynomial in $\mathbf{M}_{1,E}$ vanishes at x. Therefore, the condition of vanishing at $x \in \mathbb{P}(E)$ with multiplicity $\delta_2 - \delta_1$ defines a 1-dimensional subspace of $S^{\delta_2-\delta_1}\mathbf{S}^* \circ \mathbf{M}_{1,E}$, and (for dimensional reasons) a 2-dimensional subspace N_E of $\hat{\mathbf{M}}_{2,E}$. Hence, without loss of generality we may assume that ϕ is a polynomial vanishing at x with multiplicity $\delta_2 - \delta_1$. If we let E vary, then N_E defines a rank 2 vector subbundle $N \subset \hat{\mathbf{M}}_2 \subset S^{\delta_2}\mathbf{S}^*$. Moreover, we have $N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1) \subset S^{\delta_1}\mathbf{S}^*$. This follows from the fact that the condition of vanishing at x with multiplicity at least k defines the subbundle $\mathcal{O}_\xi(k) \oplus \cdots \oplus \mathcal{O}_\xi(\delta_2) \subset \mathcal{O}_\xi \oplus \mathcal{O}_\xi(1) \oplus \cdots \oplus \mathcal{O}_\xi(\delta_2) \cong S^{\delta_2}\mathbf{S}^*$. For every $E \in \xi$, if we choose any $0 \neq \eta \in E^*$ that vanish on $x \in \mathbb{P}(E)$, we have that the fiber of $N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1) \subset S^{\delta_1}\mathbf{S}^*$ over $E \in \xi$ is the locus of degree δ_1 polynomials ψ over $\mathbb{P}(E)$ satisfying $\eta^{\delta_2-\delta_1} \circ \psi \in \hat{\mathbf{M}}_{2,E}$.

By the description of the fibers of $\hat{\mathbf{M}}_2$ given above, all points in the zero locus of the canonical section of $q^*(N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1)) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_1)$ belong to $Z(\hat{\mathbf{s}}_2)$. Hence, the canonical section of $q^*(N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1)) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_1)$ has empty zero locus.

On the other hand, we have that $(N \otimes \mathcal{O}_\xi(-\delta_2+\delta_1))^* \otimes \mathcal{O}_\xi(\delta_1) = N^* \otimes \mathcal{O}_\xi(\delta_2)$ is a quotient of $\hat{\mathbf{M}}_2^* \otimes \mathcal{O}_\xi(\delta_2)$, so in particular it is generically ample on ξ. Then Lemma 5.1 implies that the canonical section of $q^*(N \otimes \mathcal{O}_\xi(-\delta_2+\delta_1)) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_1)$ is nonempty. Contradiction. $\square$

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